Axion-dilaton cosmology, Ricci flows and integrable structures

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In this work, based on [1], we study renormalization-group flows by deforming a class of conformal sigma-models. At leading order in α', renormalization-group equations represent a Ricci flow. In the three-sphere background, the latter is described by the Halphen system, which is exactly solvable in terms of modular forms. The round sphere is found to be the unique perturbative infra-red fixed point at one loop order.

Conformal invariance is one of the most powerful tools in string theory, allowing to go beyond the usual low-energy limits. On the other hand, it imposes stringent constraints on the theory. A concrete example is given by Wess–Zumino-Witten (WZW) models on compact groups, where the presence of the topological Wess–Zumino term gives rise to a Dirac monopole quantization (the underlying CFT is rational). The radius is therefore quantized and this creates an obstruction when trying to let it vary continuously. A possible way out would be to abandon conformal invariance. The perturbed sigma-model is no longer an infra-red fixed point and various parameters (such as the radius in the above example) run with the renormalization-group (RG) energy scale. In this work, we study the behaviour of a class of three dimensional systems living in a neighbourhood of the moduli space of the $SU(2)$ WZW model. In particular we find that their one-loop dynamics is described by a Halphen system that can be explicitly solved in terms of modular functions.

Let us consider a WZW model on a generic compact semi-simple group manifold G at level k. The standard Killing form is $\mathrm{d}s^2 = k\delta_{\alpha\beta} J^\alpha \otimes J^\beta$, where J^α are the usual left currents. We will consider a deformed WZW model with metric

$$\mathrm{d}s^2 = g_{\alpha\beta} J^\alpha \otimes J^\beta = k\gamma_\alpha(\mu)\delta_{\alpha\beta} J^\alpha \otimes J^\beta , \qquad (1)$$

where $\gamma_\alpha(\mu)$ are arbitrary positive functions and μ is the RG-scale. The B field remains unperturbed, being a topological term.

In a dimensional-regularization scheme, the one-loop RG-flow equations read [2,4,5]:

$$\frac{\mathrm{d}g_{\alpha\beta}}{\mathrm{d}\log\mu} = \frac{1}{2\pi}\left(R_{\alpha\beta} - \frac{1}{4}H^2_{\alpha\beta}\right) = \frac{1}{2\pi}R^-{}_{\alpha\beta} , \qquad (2)$$

where $R_{\alpha\beta}$ are the components of the Ricci tensor, $H = \mathrm{d}B$ and $H^2_{\alpha\beta} = H_{\alpha\gamma\delta}H_\beta{}^{\gamma\delta}$. We observe that the RG-flow equations for the perturbation pattern at hand are governed by a Ricci flow with the connection

$$\Gamma^-{}^\mu{}_{\nu\rho} = \begin{Bmatrix} \mu \\ \nu\rho \end{Bmatrix} - \frac{1}{2}H^\mu{}_{\nu\rho} . \qquad (3)$$

Although the full analysis is tractable for any compact group G, we will here focus on the $SU(2)$, where the flow equations can be solved explicitly. The metric has now only three entries: $\gamma_1(\mu), \gamma_2(\mu), \gamma_3(\mu)$. In the vielbein of the currents we obtain a diagonal Ricci tensor with entries

$$R_{11} = \frac{\gamma_1^2 - (\gamma_2 - \gamma_3)^2}{2\gamma_2\gamma_3} , \quad \text{and permutations,} \qquad (4)$$

and similarly for the Kalb–Ramond term which is diagonal and reads $\left(H^2_{11}\right) = 2/\left(\gamma_2\gamma_3\right)$ and permutations. Introduce an RG-time pointing towards the infra-red,

$$\mathrm{d}\tilde{t} = -\frac{1}{2\pi\gamma_1(\mu)\gamma_2(\mu)\gamma_3(\mu)}\mathrm{d}\log\mu , \qquad (5)$$

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where we have also reabsorbed the product of the three γ_α's. Putting everything together one obtains the following RG equations:

$$2\frac{1}{\gamma_1}\frac{\mathrm{d}\gamma_1}{\mathrm{d}\tilde{t}} = (\gamma_2-\gamma_3)^2 - \gamma_1^2 + 1\,, \quad \text{and perm.} \tag{6}$$

In the absence of torsion, the last constant term in (6) is missing and the flow converges towards a round sphere of *vanishing* radius. The presence of torsion does not alter this behaviour but affects the radius of the sphere which stabilizes to $\sqrt{k}$ because all γ_α's now converge to one. This non-trivial infra-red fixed point corresponds to the $SU(2)_k$ WZW model. Such results on the convergence of the flow are based on asymptotic analysis. However the Ricci-flow equations (6) can be solved explicitly in the case at hand. Indeed, setting $\tilde{t} = \log(T+T_0)$, and

$$\Omega_1 = \frac{\gamma_2\gamma_3}{T+T_0}\,,\ \Omega_2 = \frac{\gamma_3\gamma_1}{T+T_0}\,,\ \Omega_3 = \frac{\gamma_1\gamma_2}{T+T_0}\,, \tag{7}$$

equations (6) are recast as:

$$\frac{\mathrm{d}\Omega_1}{\mathrm{d}T} = \Omega_2\Omega_3 - \Omega_1\left(\Omega_2+\Omega_3\right)\,, \quad \text{and perm.} \tag{8}$$

This is the celebrated Halphen system that was studied in the 19th century[2]. Three different scenarios are possible:

Case 1. The case of $\Omega_1=\Omega_2=\Omega_3$ corresponds to fully isotropic deformations. In this case, the solution is unique: $\Omega_\alpha(T) = {}^1\!/_{T+A}$.

Case 2. The most general solution with $\Omega_2=\Omega_3$ is written as:

$$\begin{cases}\Omega_1(T) = \frac{1}{T+A} + \frac{C}{(T+A)^2}, \\ \Omega_2(T) = \Omega_3(T) = \frac{1}{T+A}.\end{cases} \tag{9}$$

This describes axisymmetric deformations of the three-sphere, namely deformations preserving an $SU(2)\times U(1)$.

Case 3. When $\Omega_1\neq\Omega_2\neq\Omega_3$, a class of solutions is expressed as [6]:

$$\Omega_\alpha(T) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}T}\log E_\alpha(iT), \tag{10}$$

[2]This observation was also made by K. Sfetsos in unpublished work.

where $E_\alpha(z)$ form a triplet of modular forms of weight two for $\Gamma(2)\subset PSL(2,\mathbb{Z})$ (see *e.g.* [3]). We can write the E_α as:

$$E_1 = \frac{{}^{\mathrm{d}\lambda}\!/_{\mathrm{d}z}}{\lambda}\,,\ E_2 = \frac{{}^{\mathrm{d}\lambda}\!/_{\mathrm{d}z}}{\lambda-1}\,,\ E_3 = \frac{{}^{\mathrm{d}\lambda}\!/_{\mathrm{d}z}}{\lambda(\lambda-1)}\,, \tag{11}$$

where λ is the elliptic modular form $\lambda = \vartheta_2^4/\vartheta_3^4$.

The modular properties of the functions under consideration set stringent constraints between the asymptotics of the solution and its initial conditions: large-T and small-T regimes are related by $T\leftrightarrow {}^1\!/_T$. Assuming $\Omega_\alpha^0\equiv\Omega_\alpha(0)$ *finite*, the asymptotic behaviour ($T\to\infty$) is found to be *universally*

$$\Omega_\alpha(T)\sim\frac{1}{T} + \text{subleading terms.} \tag{12}$$

This provides an elegant proof of the universality of generic $SU(2)$ Ricci flows in the presence of torsion towards the corresponding WZW infra-red fixed point.

The Ricci flow describing the renormalization of the sigma-model leads unavoidably to the round sphere, which is therefore the unique perturbative infra-red fixed point found at one loop.

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